

# NEW MAXIMUM PRINCIPLES FOR LINEAR ELLIPTIC EQUATIONS

HUNG-JU KUO<sup>†</sup> AND NEIL S. TRUDINGER<sup>‡</sup>

**ABSTRACT.** We prove extensions of the estimates of Aleksandrov and Bakel'man for linear elliptic operators in Euclidean space  $\mathbb{R}^n$  to inhomogeneous terms in  $L^q$  spaces for  $q < n$ . Our estimates depend on restrictions on the ellipticity of the operators determined by certain subcones of the positive cone. We also consider some applications to local pointwise and  $L^2$  estimates.

## 1. INTRODUCTION

In this paper, we consider linear second order partial differential operators  $L$  of the form

$$(1.1) \quad Lu := a^{ij} D_{ij} u,$$

in bounded domains  $\Omega$  in Euclidean  $n$ -space  $\mathbb{R}^n$ . The operator  $L$  is *elliptic* in  $\Omega$  if the coefficient matrix  $\mathcal{A} = [a^{ij}] : \Omega \rightarrow \mathbb{S}^n$  is positive in  $\Omega$ . Here  $\mathbb{S}^n$  denotes the linear space of  $n \times n$  real symmetric matrices and  $D^2u = [D_{ij}u] \in \mathbb{S}^n$  is the Hessian matrix of second derivatives of an appropriately smooth function  $u : \Omega \rightarrow \mathbb{R}$ . The maximum principle of Aleksandrov and Bakel'man [1, 2, 6] provides for any solution  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  of the inequalities,

$$(1.2) \quad \begin{aligned} Lu &\geq -f && \text{in } \Omega, \\ u &\leq 0 && \text{on } \partial\Omega, \end{aligned}$$

an estimate

$$(1.3) \quad \sup_{\Omega} u \leq C \left\| \frac{f}{\rho_n(\mathcal{A})} \right\|_{L^n(\Omega)}$$

where  $C$  is a constant depending on  $n$  and  $\Omega$  and the function  $\rho_n$  is given by

$$(1.4) \quad \rho_n(\mathcal{A}) = (\det \mathcal{A})^{1/n}.$$

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In the special case, where  $L$  is the Laplacian, that is  $\mathcal{A} = I$ , the exponent  $n$  in (1.3) can be improved so that

$$(1.5) \quad \sup_{\Omega} u \leq C \| f \|_{L^q(\Omega)}$$

for any  $q > n/2$ , where  $C$  is a constant depending on  $n, q$  and  $\Omega$ . Our concern in this paper is with estimates which lie between these two extreme cases. As well we shall treat more precise forms of these estimates, along with applications to local estimates.

To illustrate the nature of our results we first formulate here an extension of the estimates (1.3) (1.5). The coefficient conditions will be expressed in terms of subcones of the positive cone in  $\mathbb{S}^n$ ,  $\Gamma_n = \{\mathcal{A} \in \mathbb{S}^n \mid \mathcal{A} > 0\}$ , determined by the elementary symmetric functions  $\mathcal{S}_k$ ,  $k = 1, \dots, n$ , given by

$$(1.6) \quad \mathcal{S}_k(\lambda) = \sum \lambda_{i_1} \cdots \lambda_{i_k}$$

for  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ , where the summation is taken over all increasing  $k$ -tuples  $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ . Let us first note that for a convex symmetric cone  $\Gamma$  in  $\mathbb{R}^n$ , the dual cone  $\Gamma^*$  given by

$$(1.7) \quad \Gamma^* = \left\{ \lambda \in \mathbb{R}^n \mid \lambda \cdot \mu \geq 0, \quad \forall \mu \in \Gamma \right\}$$

is closed, convex and symmetric. We associate with the elementary symmetric function  $\mathcal{S}_k$  the open cone

$$(1.8) \quad \Gamma_k = \left\{ \lambda \in \mathbb{R}^n \mid \mathcal{S}_j(\lambda) > 0, \quad j = 1, \dots, k \right\}$$

and its closure

$$(1.9) \quad \overline{\Gamma_k} = \left\{ \lambda \in \mathbb{R}^n \mid \mathcal{S}_j(\lambda) \geq 0, \quad j = 1, \dots, k \right\}$$

which are both convex and symmetric. Clearly,  $\Gamma_k \subset \Gamma_l$  for  $k \geq l$  and  $\Gamma_1$  is the half-space,  $\Gamma_1 = \{\lambda \in \mathbb{R}^n \mid \sum \lambda_i > 0\}$ , while  $\Gamma_n$  is the positive cone,  $\Gamma_n = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, i = 1, \dots, n\}$ . Note that  $\Gamma_k$  can also be characterized as the component of the positivity set of  $\Gamma$  which contains  $\Gamma_n$ , as in [8]. Consequently the dual cones  $\Gamma_k^* \subset \Gamma_l^*$  for  $k \leq l$  with  $\Gamma_1^*$  the closed ray through  $(1, \dots, 1)$  and  $\Gamma_n^* = \overline{\Gamma_n}$ . Corresponding dual functions are determined as follows. First, we normalize  $\mathcal{S}_k$  by defining, for  $\lambda \in \Gamma_k$ ,

$$(1.10) \quad \rho_k(\lambda) = \left\{ \frac{\mathcal{S}_k(\lambda)}{\binom{n}{k}} \right\}^{1/k}.$$

We remark that the function  $\rho_k$  is increasing and concave on the cone  $\Gamma_k$ , [8], and  $\rho_k \leq \rho_l$  if  $k \geq l$ , (Maclaurin inequalities) . The dual function  $\rho_k^*$  is defined on  $\Gamma_k^*$  by

$$(1.11) \quad \rho_k^* = \inf \left\{ \frac{\lambda \cdot \mu}{n} \mid \mu \in \Gamma_k, \quad \rho_k(\mu) \geq 1 \right\}.$$

Clearly we have  $\rho_1^*(\lambda) = \ell$  where  $\lambda = \ell(1, \dots, 1)$  and  $\rho_n^*(\lambda) = \rho_n(\lambda) = (\prod \lambda_i)^{1/n}$ . As a further example, we may calculate

$$(1.12) \quad \begin{aligned} \Gamma_2^* &= \left\{ \lambda \in \mathbb{R}^n \mid |\lambda| \leq \frac{1}{\sqrt{n-1}} \sum \lambda_i \right\}, \\ \rho_2^* &= \frac{1}{\sqrt{n}} \left\{ \left( \sum \lambda_i \right)^2 - (n-1) |\lambda|^2 \right\}^{1/2}. \end{aligned}$$

We shall employ the same notation as above for matrices  $\mathcal{A} \in \mathbb{S}^n$ , writing  $\mathcal{A} \in \Gamma_k(\overline{\Gamma_k}, \Gamma_k^*)$  if the eigenvalues of  $\mathcal{A}$ ,  $\lambda = \lambda(\mathcal{A}) \in \Gamma_k(\overline{\Gamma_k}, \Gamma_k^*)$  and define  $\mathcal{S}_k(\mathcal{A}) = \mathcal{S}_k(\lambda)$ ,  $\rho_k(\mathcal{A}) = \rho_k(\lambda)$ ,  $\rho_k^*(\mathcal{A}) = \rho_k^*(\lambda)$ . We can now state the following extension of the estimates (1.3), (1.5).

**Theorem 1.1.** *Let  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy (1.2) for some coefficient matrix  $\mathcal{A} \in \Gamma_k^*$ ,  $1 \leq k \leq n$ , with  $\rho_k^*(\mathcal{A}) > 0$ . Then we have the estimate*

$$(1.13) \quad \sup_{\Omega} u \leq C \left\| \frac{f}{\rho_k^*(\mathcal{A})} \right\|_{L^q(\Omega)}$$

for  $q = k$  if  $k > n/2$  and  $q > n/2$  if  $k \leq n/2$ , where  $C$  is a constant depending on  $n, q$  and  $\Omega$ .

It follows, by approximation as in [11], that Theorem 1.1 extends to functions  $u \in W_{loc}^{2,q}(\Omega) \cap C^0(\overline{\Omega})$ . Accordingly we have the uniqueness result that if  $\mathcal{A} \in \Gamma_k^*$ ,  $\rho_k^*(\mathcal{A}) > 0$ ,  $Lu = 0$  a.e.  $(\Omega)$ ,  $u = 0$  on  $\partial\Omega$ , then  $u = 0$  in  $\Omega$ . Using the example of Gilbarg and Serrin, (see [11]),

$$(1.14) \quad Lu = \Delta u + \left( -1 + \frac{n-1}{1-\alpha} \right) \frac{x_i x_j}{|x|^2} D_{ij} u, \quad \alpha < 1$$

with solution  $u$  given by

$$(1.15) \quad u(x) = \begin{cases} |x|^\alpha & , \quad \text{if } \alpha \neq 0 \\ \log |x| & , \quad \text{if } \alpha = 0. \end{cases}$$

satisfying  $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$  if and only if  $q < \frac{n}{2-\alpha}$ , we infer that the exponent  $q$  in Theorem 1.1 cannot be improved (i.e.  $q$  can not be smaller than  $k$ ). To see this, we note, for example from [22], (see also [20]), that  $\mu = (\mu_1, \dots, \mu_n) \in \Gamma_k$  implies

$$(1.16) \quad k(n-1)\mu_i + (n-k) \sum_{j \neq i} \mu_j \geq 0$$

for any  $i = 1, \dots, n$ . For the coefficient matrix  $\mathcal{A}$  in (1.14), we then have

$$\lambda(\mathcal{A}) = \left(1, \dots, 1, \frac{n-1}{1-\alpha}\right) \in \Gamma_k^*,$$

provided  $\alpha \leq 2 - n/k$ . Consequently for  $\frac{n}{2} < k < n$ ,  $q < k$ , we choose  $\alpha = 2 - n/k$  to get a counterexample. The case  $k = n$ , follows by a slight modification, taking  $2 - n/q < \alpha < 1$  for  $q < n$ . For the case  $k \leq n/2$ , if  $q < n/2$ , we get a counterexample with  $\alpha < 0$  while for  $q = n/2$ , we may modify (1.15) by taking  $\alpha = 0$  and for  $\varepsilon > 0$ ,

$$(1.17) \quad u_\varepsilon(x) = \begin{cases} \log |x| & , \quad \text{for } |x| \geq \varepsilon \\ \frac{1}{2} \left( \frac{|x|^2}{\varepsilon^2} - 1 \right) + \log \varepsilon & , \quad \text{for } |x| < \varepsilon. \end{cases}$$

In this case  $\|Lu_\varepsilon\|_{L^q(\Omega)}$  is uniformly bounded in  $\varepsilon$  but  $\inf u_\varepsilon \rightarrow -\infty$  as  $\varepsilon \rightarrow 0$ . In this connection we mention the recent work of Astala, Iwaniec and Martin, [4], for  $n = 2$ , where an estimate of the form

$$(1.18) \quad \sup_{\Omega} u \leq C \|f\|_{L^q(\Omega)},$$

is derived for solutions of (1.2) provided  $q > \frac{2K}{K+1}$ , where  $K = \sup_{\Omega} \lambda_{\max}(\mathcal{A})/\lambda_{\min}(\mathcal{A})$  denotes the ellipticity constant of  $\mathcal{A}$ . The operator (1.14) may also be used to show that their estimate (1.18) is sharp, [4].

In the next section we will in fact prove a stronger version of Theorem 1.1, where the  $L^q$  norm is taken over the upper  $k$ -contact set of  $u$  in  $\Omega$ . In the following section, we will consider sharp versions of the estimate (1.13) in the cases  $k > n/2$ , using the Greens function from [21]. Finally in Section 4, we prove a corresponding local maximum principle and indicate the relevant extensions of other local estimates such as the Harnack and Holder estimates, [11, 12]. As an application of the local maximum principle, we obtain an extension of (1.13) for uniformly elliptic operators, with the constant  $C$  depending only on  $n, k$  and  $|\Omega|$ , analogous to [7].

Some of this paper, in particular Theorem 1.1, was proved by the second author several years ago and presented at various meetings. The two authors have also obtained

discrete versions of the case  $k = n$ , ([13, 14, 15]). It would also be interesting to have corresponding discrete versions of the estimates in this paper.

## 2. REDUCTION TO HESSIAN EQUATIONS

For  $k = 1, \dots, n$ , the  $k$ -Hessian operator  $F_k$  is defined on  $C^2(\Omega)$  by

$$(2.1) \quad \begin{aligned} F_k[u] &= \mathcal{S}_k(D^2u) \\ &= [D^2u]_k \end{aligned}$$

where for an  $n \times n$  real matrix,  $\mathcal{A}$ ,  $[\mathcal{A}]_k$  denotes the sum of its  $k \times k$  principal minors. The operator  $F_k$  is related to the linear operator in (1.1) through the following proposition.

**Proposition 2.1.** *For any matrices  $\mathcal{A} \in \Gamma_k$ ,  $\mathcal{B} \in \Gamma_k^*$ ,  $k = 1, \dots, n$ , we have the inequality,*

$$(2.2) \quad \rho_k(\mathcal{A})\rho_k^*(\mathcal{B}) \leq \frac{1}{n} \mathcal{A} \cdot \mathcal{B}.$$

**Proof.** If we fix the matrix  $\mathcal{B} = [b_{ij}]$  and minimize the inner product  $\mathcal{A} \cdot \mathcal{B}$  on the set where  $\mathcal{A} \cdot \mathcal{B} \geq 0$ ,  $\mathcal{S}_k(\mathcal{A}) = 1$ , we obtain at a critical point  $\mathcal{A}$ ,

$$\mathcal{B} = c \mathcal{D}\mathcal{S}_k(\mathcal{A})$$

for some constant  $c$ . Hence with respect to an orthonormal basis of eigenvectors of  $\mathcal{A}$ , the matrix  $\mathcal{B}$  is also diagonal with eigenvalues  $\mu = (\mu_1, \dots, \mu_n)$  given by

$$\mu_i = c D_i \mathcal{S}_k(\lambda), \quad i = 1, \dots, n,$$

where  $\lambda = \lambda(\mathcal{A})$ , [8]. The inequality (2.2) then follows from the definition (1.11).

□

Let  $u \in C^2(\Omega)$  satisfy the differential inequality (1.2). From Proposition 2.1, we then have

$$(2.3) \quad \begin{aligned} \rho_k(-D^2u)\rho_k^*(\mathcal{A}) &\leq -\frac{1}{n} Lu \\ &\leq \frac{1}{n} f \end{aligned}$$

where  $-D^2u \in \overline{\Gamma_k}$ ,  $\mathcal{A} \in \Gamma_k^*$ . Consequently, replacing  $u$  by  $-u$  we have the differential inequality,

$$(2.4) \quad F_k[u] \leq \psi, \quad \text{where} \quad \psi = \binom{n}{k} \left( \frac{f}{n \rho_k^*(\mathcal{A})} \right)^k$$

holding on the subset of  $\Omega$  where  $D^2u \in \overline{\Gamma_k}$ , that is where the function  $u$  is  $k$ -convex. The estimate (1.13) is accordingly reduced to the existence and estimation of solutions of Hessian equations, with inhomogeneous terms in  $L^p$  spaces. Indeed if  $u \in C^2(\overline{\Omega})$  is a  $k$ -convex function on  $\Omega$ , satisfying (2.4), with  $u = 0$  on  $\partial\Omega$ , then it follows readily from the Wang Sobolev inequality, [9, 24], using Moser iteration, that

$$(2.5) \quad \sup_{\Omega} u \leq C \|\psi\|_{L^p(\Omega)}^{1/k}$$

where  $p = 1$  for  $k > \frac{n}{2}$ ,  $p > \frac{n}{2k}$  for  $k \leq \frac{n}{2}$  and  $C$  depends on  $k, n, p$ , and  $\text{diam}\Omega$ .

From the estimate (2.5), we can prove Theorem 1.1, as stated, through an existence theorem for Hessian equations. Let  $\Omega_o$  be a uniformly  $(k-1)$ -convex domain, containing  $\Omega$ , with boundary  $\partial\Omega_o \in C^\infty$  and set

$$(2.6) \quad \psi' = F_k[u]\chi_{\Omega_k},$$

where  $\Omega_k = \Omega_k^-$  is the *lower k-contact set* of  $u$  in  $\Omega$  given by

$$(2.7) \quad \Omega_k^- = \left\{ x \in \Omega \mid \exists \text{ } k-\text{convex } v \in C^2(\Omega) \text{ satisfying } v \leq u \text{ in } \Omega, v(x) = u(x) \right\}.$$

Clearly for any  $x \in \Omega_k^-$ ,  $D^2u(x) \in \overline{\Gamma_k}$ . By replacing  $\Omega$  if necessary by a strictly contained subdomain we may assume  $u \in C^2(\overline{\Omega})$ , and  $\psi' \in C^o(\overline{\Omega})$ . For  $\psi_o \in C^\infty(\overline{\Omega})$  satisfying  $\psi_o > 0$  in  $\overline{\Omega}_o$ ,  $\psi' < \psi_o$  in  $\Omega$ , we define  $u_o \in C^\infty(\overline{\Omega}_o)$  to be the unique  $k$ -convex solution of the Dirichlet problem,

$$(2.8) \quad \begin{aligned} F_k[u_o] &= \psi_o && \text{in } \Omega_o, \\ u_o &= 0 && \text{on } \partial\Omega_o. \end{aligned}$$

The existence of  $u_o$  is guaranteed by the existence theorem of Caffarelli, Nirenberg and Spruck [8]; ( see also [18] ). We claim that a comparison principle holds namely  $u \geq u_0$  in  $\Omega$ . To see this we suppose there exists  $y \in \Omega$  such that

$$u_0(y) - u(y) = \sup_{\Omega}(u_0 - u) > 0.$$

Since  $u_0$  is  $k$ -convex, we must have  $y \in \Omega_k^-$ . But then  $D^2u_0(y) \leq D^2u(y)$  implies  $F_k[u_0] \leq F_k[u]$ , which contradicts (2.8). By letting  $\psi_0$  approach to  $\psi'$ , we then obtain the estimate (2.5), with  $\psi = \psi', \Omega = \Omega_0$ . Hence letting  $\Omega_k^+(u) = \Omega_k^-(u)$  denote the *upper k-contact set* of  $u$  in  $\Omega$ , we obtain the estimate

$$(2.9) \quad \sup_{\Omega} u \leq C \left\| \frac{f}{\rho_k^*(\mathcal{A})} \right\|_{L^q(\Omega_k^+)},$$

which is a more precise version of (1.13). In the next section, we shall provide a proof of (2.5) in the cases  $k > n/2$ , using [21], which leads to sharper versions of (2.9).

### 3. REFINEMENTS

In this section we refine the estimate (2.9) by using an argument analogous to that of Aleksandrov and Bakel'man for the case  $k = n$ ; see [11]. The fundamental idea is to replace cones by the graphs of the Green's functions in the cases  $k > n/2$ . For this and the following section, we need some aspects of the theory of Hessian measures developed by Trudinger and Wang in [21, 22, 23]. First we recall the general definition of  $k$ -convexity. Namely an upper semi-continuous function  $u : \Omega \rightarrow [-\infty, \infty)$  is called  $k$ -convex in  $\Omega$  if any quadratic polynomial  $p$  for which  $u - p$  has a local maximum in  $\Omega$ , satisfies  $F_k[p] \geq 0$ . General  $k$ -convex functions may be approximated by smooth ones through mollification. Indeed let us define  $\Phi^k(\Omega)$  to be the set of proper  $k$ -convex function, that is those  $\not\equiv -\infty$  on any component of  $\Omega$ . Then  $\Phi^k(\Omega) \subset \Phi_1(\Omega) \subset L_{loc}^1(\Omega)$  and the mollification  $u_h$  of  $u \in \Phi^k(\Omega)$  satisfies  $u_h \downarrow u$  as  $h \rightarrow 0$ ,  $u_h \in \Phi^k(\Omega')$  for any  $h < \text{dist}(\Omega', \partial\Omega)$ ,  $\Omega' \subset\subset \Omega$ . The main result of [21, 22] is that for any  $u \in \Phi^k(\Omega)$ , there exists a Borel measure  $\mu_k[u]$  such that

$$(3.1) \quad (i) \quad \mu_k[u](e) = \int_e F_k[u] \quad \text{for any } u \in \Phi^k(\Omega) \cap C^2(\Omega),$$

and

$$(ii) \quad \begin{aligned} &\text{if } u_m \rightarrow u \text{ a.e. in } \Omega, u_m, u \in \Phi^k(\Omega), \\ &\text{then } \mu_k[u_m] \rightarrow \mu_k[u] \text{ weakly as measures.} \end{aligned}$$

The case  $k > n/2$  is proved in [21]. Here  $\Phi^k(\Omega) \subset C^{0,2-n/k}(\Omega)$  and a.e. convergence is equivalent to uniform convergence. From the weak continuity (ii) of (3.1), it follows

that there exists a unique Green's function for  $F_k$ . That is for any point  $y \in \Omega$ , there exists a function  $G_y \in \Phi^k(\Omega)$  such that

$$(3.2) \quad \begin{aligned} \mu_k[G_y] &= \delta_y, \\ G_y &\longrightarrow 0 \text{ on } \partial\Omega. \end{aligned}$$

Here  $\delta_y$  denotes the Dirac delta measure at  $y$  and we also need to assume that  $\partial\Omega \in C^2$  is uniformly  $(k-1)$ -convex, that is the principal curvatures  $(\kappa_1, \dots, \kappa_{n-1})$  of  $\Omega$  lie in the cone  $\Gamma_{k-1}$  in  $\mathbb{R}^{n-1}$ . When  $k > n/2$ ,  $G_y \in C^{0,2-n/k}(\overline{\Omega})$ . The uniqueness is more difficult in the cases  $k \leq n/2$ , [23].

Moreover, from the interior gradient bound [19], we always have  $G_y \in C^{0,1}(\overline{\Omega} - \{y\})$ . It is easy to show, for example by smoothing the cusp, that the Green's function  $G_y$  for a ball  $B_R(y)$  of center  $y$  and radius  $R$  is given by

$$(3.3) \quad G_y(x) = \begin{cases} \frac{1}{(2 - \frac{n}{k}) \left[ \binom{n}{k} \omega_n \right]^{1/k}} (|x - y|^{2-n/k} - R^{2-n/k}) & , \text{ if } k \neq \frac{n}{2}, \\ \frac{1}{\left[ \binom{n}{k} \omega_n \right]^{1/k}} \log |x - y| & , \text{ if } k = \frac{n}{2}. \end{cases}$$

In the case  $k = n$  and convex  $\Omega$ , the Green's function  $G_y$  is the function whose graph is a cone with vertex at  $(y, G_y(y))$  and base  $\partial\Omega$ . For general  $\Omega$  and  $k > n/2$ ,  $G_y$  will have a cusp-like behavior at  $y$ , as exemplified by (3.3). It is also shown in [21], that the monotonicity property of the Monge-Ampère measure in the case  $k = n$ , extends to Hessian measures for  $k \leq n$ . Namely if  $u, v \in \Phi^k(\Omega)$  satisfy  $u \leq v$  in  $\Omega$ ,  $u = v$  continuously on  $\partial\Omega$ , then  $\mu_k[u](\Omega) \geq \mu_k[v](\Omega)$ . From this we also have a comparison principle, namely if  $u, v \in \Phi^k(\Omega)$ ,  $u \leq v$  continuously on  $\partial\Omega$ ,  $\mu_k[u] \geq \mu_k[v]$  in  $\Omega$ , then  $u \leq v$  in  $\Omega$ . As a consequence we obtain an estimate for the Greens' function, in the cases  $k > n/2$ , by comparison with (3.3). That is

$$(3.4) \quad \inf_{\Omega} G_y = G_y(y) \geq - \frac{(\text{diam } \Omega)^{2-n/k}}{(2 - \frac{n}{k}) \left[ \binom{n}{k} \omega_n \right]^{1/k}}$$

Now returning to the proof of Theorem 1.1 in Section 2, we let  $u \in \Phi^k(\Omega) \cap C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy

$$(3.5) \quad \begin{aligned} F_k[u] &\leq \psi && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where  $\partial\Omega$  is  $(k-1)$ -convex and  $k > n/2$ . Then for any point  $y \in \Omega$ , we have

$$(3.6) \quad u \leq \left[ \frac{u(y)}{G_y(y)} \right] G_y$$

since  $\mu_k[G_y] = 0$  in  $\Omega - \{y\}$ . Hence, by the monotonicity property of  $\mu_k$  and its  $k$ -homogeneity, we obtain

$$\begin{aligned} \left[ \frac{u(y)}{G_y(y)} \right]^k &= \left[ \frac{u(y)}{G_y(y)} \right]^k \mu_k[G_y](\Omega) \\ &\leq \mu_k[u](\Omega) \\ &\leq \int_{\Omega} \psi \end{aligned}$$

so that we have the precise estimate

$$(3.7) \quad \begin{aligned} -u(y) &\leq -G_y(y) \left( \int_{\Omega} \psi \right)^{1/k} \\ &\leq \frac{(\text{diam } \Omega)^{2-n/k}}{(2-n/k) \left[ \binom{n}{k} \omega_n \right]^{1/k}} \left( \int_{\Omega} \psi \right)^{1/k} \end{aligned}$$

by virtue of (3.4). Accordingly we obtain (2.9) with constant  $C$  given by

$$(3.8) \quad C = \frac{(\text{diam } \Omega)^{2-n/k}}{n(2-n/k)(\omega_n)^{1/k}}.$$

Instead of using the Green's function  $G_y$ , we may use the function

$$(3.9) \quad w_y = -\frac{G_y}{G_y(y)}$$

which can be obtained independently as the weak solution of the homogeneous Dirichlet problem,

$$(3.10) \quad \begin{aligned} F_k[w_y] &= 0 && \text{in } \Omega - \{y\}, \\ w_y &= 0 && \text{on } \partial\Omega, \\ w_y(y) &= 1, \end{aligned}$$

for example by using the Perron process. It then follows directly from (3.10) that  $w_y \in \Phi^k(\Omega) \cap C^{2-\frac{n}{k}}(\overline{\Omega}) \cap C^{0,1}(\overline{\Omega} - \{y\})$  and moreover in the estimate (3.7),

$$(3.11) \quad -u(y) \leq \left\{ \frac{1}{\mu_k[w_y]} \int_{\Omega} \psi \right\}^{\frac{1}{k}}.$$

The quantity  $\mu_k[w_y]$  is an extension to  $n/2 < k \leq n$  of the volume of the polar of  $\Omega$ , with respect to  $y$ , in the case  $k = n$ . The best constant  $C$  in (2.9) is thus given by

$$(3.12) \quad C = \frac{1}{n} \binom{n}{k}^{\frac{1}{k}} \sup_{y \in \Omega} \left\{ \mu_k[w_y] \right\}^{-\frac{1}{k}}.$$

Note that the cruder estimate (3.8) may be proved directly from (3.3) using the comparison principle.

#### 4. LOCAL ESTIMATES

In this section, we consider the full linear operator,

$$(4.1) \quad Lu := a^{ij} D_{ij} u + b^i D_i u + cu$$

under the hypothesis,  $\mathcal{A} = [a^{ij}] \in \Gamma_k^*$ , with

$$(4.2) \quad \begin{aligned} \rho_k^*(\mathcal{A}) &\geq \rho_0 \\ |\mathcal{A}| &\leq a_0 \end{aligned}$$

where  $\rho_0$  and  $a_0$  are positive constants and  $b^i, c \in L^\infty(\Omega)$ . Note that by (1.11), condition (4.2) implies  $L$  is uniformly elliptic, as on  $\Gamma_k^*$ ,

$$(4.3) \quad \begin{aligned} \lambda_n &\geq \lambda_1^{1-n} \rho_n^n(\mathcal{A}), \\ &\geq a_0^{1-n} (\rho_k^*(\mathcal{A}))^n, \\ &\geq a_0^{1-n} \rho_0^n, \end{aligned}$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  denote the eigenvalues of  $\mathcal{A}$  in decreasing order. Local pointwise estimates for  $L$  then follow from Theorem 1.1 as with the uniformly elliptic case  $k = n$ . We first consider an extension of the local maximum principle of Trudinger [11, 16].

**Theorem 4.1.** *Let  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy*

$$(4.4) \quad \begin{aligned} Lu &\geq -f && \text{in } \Omega \cap B, \\ u &\leq 0 && \text{on } \partial(\Omega \cap B), \end{aligned}$$

for some ball  $B = B_R(y) \subset \mathbb{R}^n$ ,  $f \in L^q(\Omega)$  where  $q = k$  for  $k > n/2$ ,  $q > n/2$  if  $k \leq n/2$ . Then for any concentric ball  $B_\sigma = B_{\sigma R}(y)$ ,  $0 < \sigma < 1$ , and  $p > 0$ , we have the estimate

$$(4.5) \quad \sup_{\Omega \cap B_\sigma} u \leq C \left\{ \left( R^{-n} \int_{\Omega \cap B} (u^+)^p \right)^{1/p} + \frac{R^{2-n/q}}{\rho_0} \|f\|_{L^q(\Omega \cap B)} \right\}$$

where  $C$  is a constant depending on  $\sigma, p, n$  and  $a_0/\rho_0$ ,  $\sup |b|R/\rho_0$ ,  $\sup |c|R^2/\rho_0$

*Proof.* For the proof of Theorem 4.1, we cannot directly employ the proof in [11, 16] but instead we may use that given in [17], which we indicate briefly here. First, in view of the scaling  $x \rightarrow x/R$ ,  $f \rightarrow f/R^2$ , we may take  $R = 1$ ,  $y = 0$ . Let

$$(4.6) \quad \eta = [(1 - |x|^2)^+]^\beta$$

for  $\beta \geq 1$ , to be chosen. Setting  $v = \eta(u^+)^2$ , we compute in  $\Omega \cap B \cap \{u > 0\}$

$$(4.7) \quad \begin{aligned} L_0 v &:= a^{ij} D_{ij} v \\ &= (u^+)^2 a^{ij} D_{ij} \eta + 2a^{ij} D_i \eta D_j u^2 + \eta a^{ij} D_{ij} u^2 \\ &\geq -C(|\mathcal{A}| + |b| + |c|) \eta^{1-2/\beta} u^2 + 2\eta u f \end{aligned}$$

where  $C$  depends on  $n, \lambda_1/\lambda_n$  and  $\beta$ . Now following [17], we apply Theorem 1.1, to obtain

$$(4.8) \quad \sup_{\Omega \cap B_\sigma} v \leq C \left\{ \|v^{1-2/\beta} (u^+)^{4/\beta}\|_{L^q(B \cup \Omega)} + \frac{1}{\rho_0} \|v^{1/2} f\|_{L^q(B \cup \Omega)} \right\},$$

from which we deduce (4.5), by taking  $\beta = 4q/p$ .  $\square$

We remark that we may only assume  $b \in L^{2q}(\Omega)$ ,  $c \in L^q(\Omega)$  in Theorem 4.1. Also, as remarked in [17] for the case  $k = n$ , we obtain by choosing  $R$  sufficiently large in Theorem 4.1, the following variant of the Aleksandrov-Bakel'man principle for uniformly elliptic operators.

**Corollary 4.2.** *Let  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy (1.1), (1.2) under the above hypothesis on  $L$ . Then we have the estimate*

$$(4.9) \quad \sup_{\Omega} u \leq C |\Omega|^{2/n-1/q} \left\| \frac{f}{\rho_k^*(\mathcal{A})} \right\|_{L^q(\Omega)},$$

where  $q = k$  if  $k > n/2$ ,  $q > n/2$  if  $k \leq n/2$  and  $C$  is a constant depending on  $n, q, a_0/\rho_0$ .

The special case  $k = n$  of (4.9) was found differently by Cabré in [7].

For solutions, the *Hölder* and *Harnack* estimates of Krylov and Safonov, (see [11, 12, 16, 17], extend automatically to inhomogeneous terms in lower  $L^p$  spaces. This is readily seen, for example by following the proof in [11, 16].

**Theorem 4.3.** *Let  $u \in C^2(\Omega)$  satisfy  $Lu = f$  in  $B = B_R(y) \subset \Omega$ . Then for any concentric ball  $B_\sigma = B_{\sigma R}(y)$ ,  $0 < \sigma < 1$ , we have the estimate*

$$(4.10) \quad \text{osc}_{B_\sigma} u \leq C \sigma^\alpha \left\{ \text{osc}_B u + R^{2-n/q} \left\| \frac{f}{\rho_k^*(\mathcal{A})} \right\|_{L^q(B)} \right\}$$

where  $q$  is as in Theorem 4.2,  $\alpha > 0$  depends on  $n, a_0/\rho_0$  and  $C$  depends on  $n, q, a_0/\rho_0, \sup |b|R/\rho_0, \sup |c|R^2\rho_0$ . Furthermore if  $u \geq 0$  in  $B$ , then for any  $0 < \sigma, \tau < 1$ ,

$$(4.11) \quad \sup_{B_\sigma} u \leq C \left\{ \inf_{B_\tau} u + R^{2-n/q} \left\| \frac{f}{\rho_k^*(\mathcal{A})} \right\|_{L^q(B)} \right\}$$

where  $C$  depends on the same quantities as in (4.10) together with  $\sigma$  and  $\tau$ .

We remark that as in Theorem 4.1 we need only assume  $b/\rho_0 \in L^{2q}(B), c/\rho_0 \in L^q(B)$ . Other local estimates which depend on the Aleksandrov-Bakel'man maximum principle also extend in a corresponding way.

Finally we remark on an interesting relationship between the case  $k = 2$  and second derivative estimates. Indeed we first note another characterization of  $\Gamma_2^*$ , namely

$$(4.12) \quad \Gamma_2^* = \left\{ \lambda \in \mathbb{R}^n \mid \left\| \frac{(n-1)}{\text{tr}\mathcal{A}} \mathcal{A} - I \right\|_2 < 1 \right\}$$

Consequently, we have by perturbation from the case  $L = \Delta$ , (see [11]), that if  $u \in C^2(\Omega)$ ,  $Lu = f$  in  $\Omega$  and (4.2) holds for  $k = 2$  then for any  $\Omega' \subset\subset \Omega$ ,

$$(4.13) \quad \|D^2u\|_{L^2(\Omega')} \leq C \left\{ \|u\|_{L^2(\Omega)} + \left\| \frac{f}{\rho_2^*(\mathcal{A})} \right\|_{L^2(\Omega)} \right\}$$

where  $C$  depends on  $n$ ,  $\Omega$ ,  $\Omega'$ ,  $a_0/\rho_0$ ,  $b_0/\rho_0$  and  $c_0/\rho_0$ . Now if we take  $n = 3$  and apply Corollary 4.2, we obtain the full estimate

$$(4.14) \quad \|u\|_{W^{2,2}(\Omega')} \leq C \left\| \frac{f}{\rho_2^*(\mathcal{A})} \right\|_{L^2(\Omega)}.$$

if  $L$  is of the form (1.1) and  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$  vanishes on  $\partial\Omega$ . If  $\partial\Omega \in C^{1,1}$ , then we may replace  $\Omega'$  by  $\Omega$ .

Note that the estimates of this section also extend by approximation to functions  $u$  in Sobolev spaces  $W_{\text{loc}}^{2,q}(\Omega)$ .

## REFERENCES

- [1] A.D. Aleksandrov, *Certain estimates for the Dirichlet problem*, Dokl. Akad. Nauk. SSSR, **134**(1960), 1001-1004, [Russian], English translation in Soviet Math. Dokl. **1**(1960), 1151-1154.
- [2] A.D. Aleksandrov, *Uniqueness conditions and estimates for the solution of the Dirichlet problem*, Vestnik Leningrad Univ. **18**(1966), 5-29, [Russian], English transl., Amer. Math. Soc. Transl. (2) **68**(1968), 89-119.
- [3] A.D. Aleksandrov, *Majorization of solutions of second-order linear equations*, Vestnik Leningrad Univ. **21**(1996), 5-25, [Russian], English translation in Amer. Math. Soc. Transl. (2) **68**(1968), 120-143.
- [4] K. Astala, T. Iwaniec, G. Martin, *Pucci's conjecture and the Aleksandrov inequality for elliptic PDEs in the plane*, J. reine angew Math. **591**(2006), 49-74.
- [5] I. Ya. Bakeman, *Theory of quasilinear elliptic equations*, Sibirsk. Mat. Zh. **2**(1961), 179-186. [Russian]
- [6] I. Ya. Bakeman, *Convex Analysis and Nonlinear Geometric Elliptic Equations*, Springer-Verlag, 1994.
- [7] X. Cabré, *On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math. **49**(1996).
- [8] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations III : Functions of the eigenvalues of the Hessian*, Acta Math. **155**(1985) 261-301.
- [9] K.S. Chou and X.-J. Wang, *Variational theory for Hessian equations*, Comm. Pure Appl. Math., **54**(2001), 1029-1064.
- [10] G. Fabes and D. Strook, *The  $L^p$  integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. **41**(1984), 977-1016.
- [11] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edition, Springer Verlag, 1983.

- [12] N.Y. Krylov and M. V. Safonov, *Certain properties of solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk SSSR **44**(1980) 161-175, [Russian], English translation in Math. USSR-Izv. **16**(1981), 151-164.
- [13] H. J. Kuo and N. S. Trudinger, *Positive difference operators on general meshes*, Duke Math. J. **83**(1996), 415-433.
- [14] —, *Maximum principles for difference operators*, Topics in Partial Differential Equations & Applications: Collected papers in Honor of Carlo Pucci, Lecture Notes in Pure and Applied Mathematics Series/**177**, 209-219, Marcel Dekker, Inc. 1996.
- [15] —, *A note on the discrete Aleksandrov-Bakelman maximum principle*, Taiwanese J. of Math., **4**(2000), 55-67.
- [16] N.S. Trudinger, *Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations*, Invent. Math., **61**(1980), 67-79.
- [17] —, *Lectures on Nonlinear Equations of Second Order*, Lectures in Mathematical Sciences, University of Tokyo, New Series Vol. 9, (1995).
- [18] —, *On the Dirichlet problem for Hessian equations*, Acta Math., **175**(1995), 151-164.
- [19] —, *Weak solutions of Hessian equations*, Comm. PDE, **22**(1997), 1251-1261.
- [20] —, *On Hessian Measures for non-commuting vector fields*, Q. J. Pure Appl. Math., **2**(2006), 149-163.
- [21] N.S. Trudinger and X.-J. Wang, *Hessian measures I*, Topol. Methods Nonlin. Anal., **10** (1997), 225-239.
- [22] —, *Hessian measures II*, Ann. Math., **150**(1999), 579-604.
- [23] —, *Hessian measures III*, J. of Funct. Anal., **193**(2002), 1-23.
- [24] X.-J. Wang, *A class of fully nonlinear elliptic equations and related functionals*, Indiana Univ. Math. J., **43**(1994), 25-54.

<sup>†</sup>DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHUNG-HSING UNIVERSITY, TAICHUNG 402, TAIWAN.

*E-mail address:* kuohj@nchu.edu.tw

<sup>‡</sup>CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 0200, AUSTRALIA.

*E-mail address:* neil.trudinger@anu.edu.au